LECTURE 28 CONCAVITY AND CURVE SKETCHING

We finish the last example from last class.

Example. On $[0, 2\pi]$, find the critical points of $g(x) = \sin^2(x) - \sin(x) - 1$, identify the open intervals on which f is increasing and on which f is decreasing. Determine the extrema.

Solution. The critical points satisfy

$$0 = g'(x) = 2\sin(x)\cos(x) - \cos(x) \implies \cos(x)(2\sin(x) - 1) = 0$$

which implies

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}.$$

So, the intervals we form is

$$\left(0, \frac{\pi}{6}\right), \left(\frac{\pi}{6}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{5\pi}{6}\right), \left(\frac{5\pi}{6}, \frac{3\pi}{2}\right), \left(\frac{3\pi}{2}, 2\pi\right).$$

We find $-, +, -, +, -$ respectively, of the signs of $f'(x)$. Thus, local max at $x = \frac{\pi}{6}, \frac{3\pi}{2}$, local min at $x = \frac{\pi}{2}, \frac{5\pi}{6}$. Now, for absolute extrema, we include the endpoints,

$$g(0) = -1, \quad g(2\pi) = -1$$

while

$$g\left(\frac{\pi}{6}\right) = \frac{1}{4} - \frac{1}{2} - 1 = -\frac{5}{4}, \quad g\left(\frac{\pi}{2}\right) = -1, \quad g\left(\frac{5\pi}{6}\right) = -\frac{5}{4}, \quad g\left(\frac{3\pi}{2}\right) = 1.$$

The absolute maximum is at $g\left(\frac{3\pi}{2}\right) = 1$ and the absolute minimum is at $g\left(\frac{\pi}{6}\right) = -\frac{5}{4}$ and $g\left(\frac{5\pi}{6}\right) = -\frac{5}{4}$.

The Second Dervative and Concavity

There are two ways of increasing, $\curlyvee$ or $\curlywedge$. The difference here is how it curves, or whether the curve faces down or up.

Definition. The graph of a differentiable function $y = f(x)$ is

(1) **concave up/convex** on an open interval I if f' is increasing on I.
(2) **concave down** on an open interval I if f' is decreasing on I.

If f'' exists, then we can use the corollary on monotone functions here.

The second dervative test for concavity.

Let $f(x)$ be a twice-differentiable function on some interval I.

(1) If $f'' > 0$ on I, then the graph of f on I is **concave up/convex**.
(2) If $f'' < 0$ on I, then the graph of f on I is **concave down**.

Example. Determine the concavity of $y = 3 + \sin(x)$ on $[0, 2\pi]$.

Solution. First, note that we have a trigonometric function, which is infinitely differentiable. Hence, we can use the second derivative test for concavity. We compute the 2nd derivative and find $y''(x) = -\sin(x)$. Now, we ask, for which x is $y'' > 0$ and $y'' < 0$ respectively. On the one hand,

$$y'' = -\sin(x) > 0 \implies \sin(x) < 0 \implies \pi < x < 2\pi;$$

on the other hand,

$$y'' = -\sin(x) < 0 \implies \sin(x) > 0 \implies 0 < x < \pi.$$

This means, $y = 3 + \sin(x)$ is concave up on $(\pi, 2\pi)$ and concave down on $(0, \pi)$.

Note from the previous example that at $(\pi, 3)$, the concavity of y changes from concave down to concave up as x increases past π. We learned about critical points ($f' = 0$). How about $f'' = 0$? This point physically means that the concavity is changing.

Definition. A point $(c, f(c))$ where the graph of a function has a tangent line and the concavity changes is a **point of inflection**.

Proposition. *If f is twice differentiable, then at the point of inflection, we must have $f'' = 0$ or undefined.*

Note that this statement only goes one way – it is a necessary condition for the point of inflection, not sufficient. When you have $f'' = 0$ at some point, it does **NOT** guarantee that the point is a point of inflection. It **COULD** be, but doesn't have to be. The fact that it could be means solving $f'' = 0$ is still valuable.

Simple example to showcase the insufficiency.

Example. $f(x) = x^4$. Certainly, $f''(0) = 0$. But $f''(x) = 12x^2$ is positive on both sides of $x = 0$, which means there is no concavity change! Hence $(0, 0)$ is NOT an inflection point.

Remark. **MORAL OF THE STORY**: you confirm inflection point by looking at the left and right of the point where $f'' = 0$. If there is a sign change of f'', then inflection point is confirmed.

Example. Consider $f(x) = x^{5/3}$. It does have a flat tangent line at $x = 0$, namely, $f'(x) = \frac{5}{3}x^{\frac{2}{3}} \implies f'(0) = 0$. However, the second derivative $f''(x) = \frac{10}{9}x^{-\frac{1}{3}}$ is undefined at 0. This COULD make $x = 0$ an inflection point (suspect). Indeed, we find that $f''(x) = \frac{10}{9}x^{-1/3} < 0$ if $x < 0$ and $f''(x) = \frac{10}{9}x^{-1/3} > 0$ if $x > 0$, which means f'' has a sign change past $x = 0$. Therefore, $x = 0$ is an inflection point.

Example. Consider the particle motion governed by the position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \quad t \geq 0.$$

The velocity is

$$v(t) = s'(t) = 6t^2 - 28t + 22$$

where the critical points

$$0 = v(t) = 2\left(3t^2 - 14t + 11\right) = 2(3t - 11)(t - 1) \implies t = \frac{11}{3}, \quad t = 1.$$

The intervals are $(0, 1)$, $\left(1, \frac{11}{3}\right)$ and $\left(\frac{11}{3}, \infty\right)$. We find increasing, decreasing and increasing respectively.

The acceleration is

$$a(t) = 12t - 28$$

where we check concavity changes. We find that $t = \frac{7}{3}$ is the "critical turning point" in velocity, and we check the sign of $a(t)$ on the left and right. We find $a < 0$ for $t \in \left(0, \frac{7}{3}\right)$ and $a > 0$ for $t \in \left(\frac{7}{3}, \infty\right)$. The graph of s then is concave up on $\left(0, \frac{7}{3}\right)$ and concave down on $\left(\frac{7}{3}, \infty\right)$.

Theorem. *(Second derivative test for local extrema)*

Suppose f'' is continuous on an open interval containing $x = c$.

1. *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.*
2. *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.*
3. *If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.*

Example. Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

using the following steps

1. Identify the domain of f and symmetries the curve may have.
2. Find the derivatives y' and y''.
3. Find the critical points of y, if any, and identify the function's behaviour at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes that may exist.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve together with any asymptotes that exist.